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ON THE MEAN CONTINUITY OF GAUSS

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The purpose of this paper is to introduce a new notion which we call the *mean continuity principle* for second order uniformly elliptic partial differential operators on manifolds and to propose the study of characterizing those operators satisfying the mean continuity principle. This new notion is expected to play an important and essential role in discussing one direction of the Weyl lemma for the above operators concerning the continuity of distributional solutions, not the hypoellipticity of them. First we explain how we come to this notion while we are studying the relation between distributional and axiomatic definitions of superharmonicity for stationary Schrödinger operators on the Euclidean regions. After giving the definition of this new notion we formulate the problem when second order elliptic operators on manifolds satisfy this principle of mean continuity. We then report a few results on this problem in the starting stage of our investigations as samples of possible results in this direction. Roughly speaking the operators satisfy the mean continuity principle if the coefficients of operators are sufficiently smooth. Thus the problem should be considered in future from the view point that how much the regularity of coefficients of elliptic operators can be weakened.

1. Motivation. Consider Schrödinger operators $-\Delta+\mu$ on a subregion M in the d dimensional Euclidean space $\mathbb{R}^d$ $(d \geqq 2)$ with potentials μ in the family $\mathcal{K}(M)$ of signed Radon measures of Kato class on M characterized by

$$(1.1) \qquad \lim_{r\downarrow 0}\left(\sup_{|x-c|<r}\int_{|y-c|<r} N(x-y)d|\mu|(y)\right)=0$$

for every point $c \in M$ with $N(t) = \log(1/|t|)$ $(d = 2)$ and $1/|t|^{d-2}$ $(d \geqq 3)$. Since the notion of Kato measures is locally defined one, the family $\mathcal{K}(M)$ can be defined in the same fashion as above even for C^1 manifolds M by considering in

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each parametric neighborhood of M. Set

$$^dH_\mu(M) := \{u \in L^1_{loc}(M, \lambda + |\mu|) : (-\Delta + \mu)u = 0)\},$$

where λ is the Lebesgue measure on $\mathbb{R}^d$ and $|\mu|$ is the total variation measure of μ, and we call members u in the class

$$H_\mu(M) := {}^dH_\mu(M) \cap C(M)$$

μ *harmonic* functions on M. As generalization of the classical Weyl lemma we proved ([10], see also [3]), on setting $\mu = (d\mu/d\lambda)\lambda + \mu_s$, that

$$H_\mu(M) = {}^dH_\mu(M) \cap C(\mathrm{supp}\ \mu_s; M). \tag{1.2}$$

Here $C(X; M)$ $(X \subset M)$ is the class of functions on M continuous at each point of X with $C(M; M) = C(M)$ and supp μ_s is the support of the singular part μ_s of μ. The theorem (1.2) is a consequence of the following more precise fact ([10]): *$u \in {}^dH_\mu(M)$ belongs to $H_\mu(M)$ if and only if u is continuous μ_s almost everywhere on M.*

As the counterpart of $^dH_\mu(M)$ set

$$^dS_\mu(M) := \{u \in L^1_{loc}(M, \lambda + |\mu|) : (-\Delta + \mu)u \geqq 0\}$$

and as the counterpart of $H_\mu(M)$ set

$$S_\mu(M) := \{u : \mu \text{ superharmonic on } M\},$$

where u is μ *superharmonic* on M if $u \in \mathrm{lsc}(M)$, $u > -\infty$ on M, $u \not\equiv +\infty$ on M, and u is harmonically concave, i.e. $u \geqq (H_\mu)^V_u$ on every small ball $V \subset M$, where $(H_\mu)^V_u$ is the Perron-Wiener-Brelot solution of the Dirichlet problem concerning H_μ on the ball V with the boundary data u on ∂V. Here we mean by $\mathrm{lsc}(X; M)$ for $X \subset M$ the class of functions on M lower semicontinuous at each point of X with $\mathrm{lsc}(M; M) = \mathrm{lsc}(M)$. It may be impressive to call u in $^dS_\mu(M)$ ($S_\mu(M)$, resp.) *distributionally* (*axiomatically*, resp.) μ superharmonic.

We felt it was quite a natural guess based upon (1.2) that the Weyl lemma for superharmonic functions should be

$$S_\mu(M) = {}^dS_\mu(M) \cap \mathrm{lsc}(\mathrm{supp}\ \mu_s; M)$$

and therefore it was a surprise for us to find a counterexample to the above expectation. We found ([11]) the true Weyl lemma in superharmonic version is

$$S_\mu(M) = {}^dS_\mu(M) \cap \mathrm{mc}(\mathrm{supp}\ \mu_s; M), \tag{1.3}$$

where $\mathrm{mc}(X;M)$ $(X \subset M)$, with $\mathrm{mc}(M;M) = \mathrm{mc}(M)$, is the class of functions f on M *mean continuous* at each point $x_0 \in X$ in the sense that

$$\lim_{r\downarrow 0} \frac{1}{\lambda(B(x_0,r))} \int_{B(x_0,r)} f(x)d\lambda(x) = f(x_0), \tag{1.4}$$

where $B(x_0,r)$ is the ball with radius $r > 0$ centered at x_0. The meaning of (1.3) is, first of all, $S_\mu(M) \subset {}^dS_\mu(M)$ (the Riesz decomposition theorem) and then a $u \in {}^dS_\mu(M)$ belongs to $S_\mu(M)$ if and only if u is mean continuous at each point of supp μ_s so that $S_\mu(M) = {}^dS_\mu(M)$ (the classical Weyl lemma) if and only if μ is absolutely continuous. Actually the theorem (1.3) follows from the following more precise assertion: *$u \in {}^dS_\mu(M)$ belongs to $S_\mu(M)$ if and only if u is mean continuous μ_s almost everywhere on M.* The essence of (1.3) lies in the classical Gauss mean continuity property that $S_0(M) \subset \mathrm{mc}(M)$ ($S_\mu(M) \subset \mathrm{mc}(M)$ in general), which follows from the Gauss mean value theorem for $H_0(M)$. To obtain (1.3) the condition (1.4) ensuring the mean continuity of $S_\mu(M)$ is more than sufficient and we may weaken the definition of mean continuity as follows: f is *mean continuous* at x_0 if there is a decreasing net $(V_\iota)_{\iota\in I}$ (with I a directed set) of neighborhoods V_ι of x_0 with $\bigcap_{\iota\in I} V_\iota = \{x_0\}$ and a measure v on $\bigcup_{\iota\in I} V_\iota$ comparable to λ such that

$$\lim_{\iota} \frac{1}{v(V_\iota)} \int_{V_\iota} f(x)dv(x) = f(x_0). \tag{1.5}$$

These observations mentioned thus far led us to study the mean continuity for general elliptic operators.

2. Problem. Let M be an orientable and connected manifold of class C^∞ whose dimension $d \geqq 2$. The local coordinate of a point $x \in M$ is denoted by $(x^1, x^2, \cdots, x^d)$. Following the convention of the tensor analysis we use the Einstein convention: whenever an index $i \in \{1, \cdots, d\}$ appears both in the upper and lower position, it is understood that summation for $i = 1, \cdots, d$ is carried out (cf. e.g. [12]). We fix an elliptic partial differential operator A on M given by

$$Au(x) := \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \left(\sqrt{a(x)} a^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right) + b^i(x) \frac{\partial u(x)}{\partial x^i}, \tag{2.1}$$

where $(a^{ij}(x))$ is a C^∞ contravariant tensor of order 2 and a strictly positive symmetric matrix at each point $x \in M$, $(b^i(x))$ is a contravariant vector which is also of class C^∞ in each parametric ball, and $a(x) := \det(a_{ij}(x))$ with $(a_{ij}(x)) := (a^{ij}(x))^{-1}$ (the inverse matrix of $(a^{ij}(x))$).

It is convenient to introduce the Riemannian metric ds by

$$ds^2 = a_{ij} dx^i dx^j = (dx^i)(a^{ij}(x))^{-1\,t}(dx^i)$$

on M, where (dx^i) is $1 \times d$ matrix $(dx^1, \cdots, dx^d)$ and ${}^t(dx^i)$ is the transposed $d \times 1$ matrix of (dx^i), so that $M = (M, ds)$ is a Riemannian manifold of class C^∞ and

$$A = \Delta_M + b^i(x)\frac{\partial}{\partial x^i},$$

where $-\Delta_M$ is the proper Laplace-Beltrami operator $d\delta + \delta d$ on $M = (M, ds)$.

Fixing A on M we consider the Schrödinger type operator $-A + \mu$ on M with its potential μ in the family $\mathcal{K}(M)$ of *Kato measures* μ, where $\mathcal{K}(M)$ is the family of signed Radon measures μ on M satisfying (1.1) in each parametric ball in M. We denote by λ the proper volume measure on (M, ds), i.e.

$$d\lambda(x) := \sqrt{a(x)}dx^1 \cdots dx^d.$$

The λ is a typical member of $\mathcal{K}(M)$. We denote by

$$H_\mu(U) = H_{-A+\mu}(U) := \{u \in L^1_{loc}(U, \lambda + |\mu|) : (-A + \mu)u = 0\} \cap C(U)$$

for each open subset U of M. A function u on U is said to be μ *harmonic* (or, more precisely, $(-A+\mu)$ harmonic if indication of $-A$ is important). Then $U \mapsto H_\mu(U)$ gives a harmonic sheaf satisfying the Brelot axioms so that M with the harmonic structure H_μ: (M, H_μ) is a Brelot (harmonic) space (cf. e.g. [1]). Therefore the notion of superharmonic functions on M can be considered and we denote by

$$S_\mu(M) = S_{-A+\mu}(M) := \{u : u \text{ is } \mu \text{ superharmonic on } M\}, \tag{2.2}$$

where a function u on M is μ *superharmonic* (or more precisely, $(-A+\mu)$ superharmonic) on M if the following 3 conditions are satisfied: u is lower semicontinuous on M; $u > -\infty$ on M and $u \not\equiv +\infty$ on M; u is μ harmonically concave on M, i.e. $(H_\mu)_u^V \leqq u$ on V for every small Euclidean balls V considered in each parametric ball of M, where $(H_\mu)_u^V$ is the Perron-Wiener-Brelot solution of the Dirichlet problem concerning H_μ on V with boundary data u on ∂V.

We denote by $B(x, r)$ the *geodesic ball* on (M, ds), i.e. the set of points $y \in M$ whose geodesic distance from x is less than $r > 0$. Here we only consider so small $r > 0$ that there is a C^∞ diffeomorphism φ of the closure $\overline{B}(x, r)$ of $B(x, r)$ to a closed Euclidean ball $\overline{E}$ such that $\varphi | B(x, r)$ sends $B(x, r)$ to the interior E of $\overline{E}$. A function f defined on M is said to be *mean continuous* on M at a point $x_0 \in M$ with respect to the tensor (a_{ij}) (or the metric ds) if

$$\lim_{r \downarrow 0} \frac{1}{\lambda(B(x_0, r))} \int_{B(x_0, r)} f(x) d\lambda(x) = f(x_0). \tag{2.3}$$

We denote the set $\{f : f \text{ is mean continuous on } (M, ds)\}$ by

$$\mathrm{mc}(M) = \mathrm{mc}(M, (a_{ij})) = \mathrm{mc}(M, ds). \tag{2.4}$$

We say that the operator $-A+\mu$ satisfies the *mean continuity principle* if

$$S_\mu(M) \subset \mathrm{mc}(M), \tag{2.5}$$

or more precisely $S_{-A+\mu}(M) \subset \mathrm{mc}(M, (a_{ij}))$. In the case of Motivation in section 1, we had the same relation as (2.5) above, but precisely speaking $S_{-\Delta+\mu}(M) \subset \mathrm{mc}(M, (\delta_{ij}))$, so that $-\Delta + \mu \quad (\mu \in \mathcal{K}(M))$ on any Euclidean region M satisfies the mean continuity principle. Then we now ask:

PROBLEM. *Does every $-A+\mu$ with potential $\mu \in \mathcal{K}(M)$ satisfy the mean continuity principle?*

3. Result. By virtue of the fact that we are assuming sufficient regularity of coefficients of the Schrödinger type operator $-A+\mu$ (i.e. tensors $(a^{ij}(x))_{1\leqq i,j\leqq d}$ and $(b^i(x))_{1\leqq i\leqq d}$ in the strictly elliptic operator A given by (2.1) are of class C^∞ and the potential μ of the operator $-A+\mu$ is of Kato class), we can give an affirmative answer to the problem stated at the end of the foregoing section 2.

THEOREM 3.1. *The operator $-A+\mu$ on M satisfies the mean continuity principle, i.e. for every $(-A+\mu)$ superharmonic function u defined in a vicinity of an arbitrarily given point x_0 in M, the following relation holds:*

$$\lim_{r\downarrow 0} \frac{1}{\lambda(B(x_0,r))} \int_{B(x_0,r)} u(x)d\lambda(x) = u(x_0), \tag{3.1}$$

where $B(x_0,r)$ is the geodesic ball with radius r centered at x_0 on the Riemannian manifold (M, ds) and $d\lambda$ the volume element on (M, ds) with ds the metric on M induced by $(a^{ij}(x))^{-1}$.

We gather from the proof of the above theorem which will be given below that the regularity of coefficients of A may be weakened at least to the extent that $(a^{ij}(x))$ is of class C^2 and $(b^i(x))$ is locally Hölder continuous but presently it is merely our guess. The proof of the above theorem 3.1 goes as follows. First we prove

REDUCTION 3.2. *If the mean continuity principle is valid for the operator $-A$ on M, then the operator $-A+\mu$ satisfies the mean continuity principle for any potential μ in $\mathcal{K}(M)$.*

After giving the proof of the above in the section 4, we prove that $-A$ satisfies the mean continuity principle in §5 , which completes the proof of Theorem 3.1. By choosing $(b^i(x)) = 0$ in (2.1) we have the following direct consequence of Theorem 3.1. Actually to clarify this was the original incentive to our present study.

COROLLARY 3.3. *Let $-\Delta_M$ be the proper Laplace-Beltrami operator*

$$-\Delta_M u(x) = -\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{a(x)}a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right)$$

on the Riemannian manifold (M, ds) given by $ds^2 = a_{ij}(x)dx^i dx^j$ with $(a_{ij}(x)) = (a^{ij}(x))^{-1}$ and $a(x) = \det(a_{ij}(x))$ and μ be any Kato measure on M. Then the operator $-\Delta_M + \mu$ satisfies the mean continuity principle:

$$S_{-\Delta_M+\mu}(M) \subset \mathrm{mc}(M, ds). \tag{3.2}$$

4. **Proof of Reduction 3.2.** The sheaf $H_A = H$ of genuine solutions u of $-Au = 0$ in the classical sense is a harmonic sheaf which gives rise to a Brelot space (cf. e.g. [2]). Every regular subregion W of M has a *Green function* $g_A^W(\cdot, y) = g^W(\cdot, y)$ with its pole at y, which is a minimal positive solution of a Poisson equation

$$-Ag^W(\cdot, y) = \delta_y \tag{4.1}$$

on W, where δ_y is the Dirac measure supported by y. The Green function $g^W(\cdot, y)$ may also be characterized by the following two properties (a) and (b) (cf. e.g. [9]): (a) $g^W(\cdot, y)$ is a potential on W for every $y \in W$ and its support is $\{y\}$; (b) For any $y \in W$ we have

$$\lim_{x\to y}\frac{g^W(x,y)}{\left(\sqrt{(x-y)(a^{ij}(x))^t(x-y)}\right)^{2-d}} = \sqrt{a(y)} \tag{4.2}$$

for $d \geqq 3$ and the denominator of the fraction under the limit on the left hand side of the above formula must be replaced by $\log\left(\sqrt{(x-y)(a^{ij}(x))^t(x-y)}\right)^{-1}$ for $d = 2$. The function $g^W(x, \cdot)$ is the Green function with its pole at $x \in W$ for the (formal) adjoint operator A^* of A. In view of (4.2) we see that

$$\lim_{r\downarrow 0}\left(\sup_{x\in B(c,r)}\int_{B(c,r)} g^W(x,y)d|\mu|(y)\right) = 0 \tag{4.3}$$

for every $c \in W$ if $\mu \in \mathcal{K}(M)$.

Choose arbitrarily and then fix a point $c \in M$ and a regular geodesic ball $B(c, R)$ on M. By embedding $B(c, R)$ in $\mathbb{R}^d$ we can consider the kernel $N(x-y)$ for x and y in $B(c, R)$ (see (1.1)). Fix a $0 < \rho < R$ such that $x-y \in B(c, R)$ and $N(x-y) > 0$ if x and y are in $B(c, \rho)$. For each $c \in M$ we fix $0 < \rho < R$ as above. Then there exists a $\gamma = \gamma(c) \in (0,\infty)$ independent of the choice of $V = B(c,r)$ $(0 < r < \rho)$ such that the following *3G inequality* holds:

$$\frac{g^V(x,z)g^V(z,y)}{g^V(x,y)} \leqq \gamma(c)(N(x-z)+N(z-y)) \tag{4.4}$$

for any triple (x,y,z) of three points in V (cf. e.g. [5] and [1]). We may call the smallest possible $\gamma = \gamma(c)$ the *3G constant* at $c \in M$ with respect to A such that (4.4) holds for any sufficiently small geodesic ball V centered at c.

For any fixed $\mu \in \mathcal{K}(M)$, we denote by

$$^dH_\mu(U) = {}^dH_{-A+\mu}(U) := \{u \in L^1_{loc}(U, \lambda + |\mu|) : (-A+\mu)u = 0\}$$

for each open subset $U \subset M$ and

$$H_\mu(U) = H_{-A+\mu}(U) := {}^dH_{-A+\mu}(U) \cap C(U).$$

Then $H_\mu = H_{-A+\mu}: U \mapsto H_\mu(U)$ is a harmonic sheaf which gives rise to a Brelot space ([1]) and we will call each $u \in H_\mu(U)$ μ *harmonic*, or more precisely, $(-A+\mu)$ harmonic on U.

Consider an operator $T = T^V = T^V_\mu$ with $V := B(c,r)$ sufficiently small by

$$(Tf)(x) = \int_V g^V(x,y)f(y)d\mu(y) \tag{4.5}$$

from $L^1(V, \lambda + |\mu|)$ to itself, which can be seen by using the Fubini theorem and noting the Kato property of μ. We define $|T|$ from T by

$$(|T|f)(x) = \int_V g^V(x,y)f(y)d|\mu|(y).$$

Fix any $0 < q < 1/2$ and choose and then fix an $r > 0$ so small that

$$\gamma(c)\int_V (N(x-z)+N(z-y))d|\mu|(z) < q \quad (V := B(c,r)). \tag{4.6}$$

Then we can solve the integral equation $(I+T)u = g^V(\cdot,y)$ on V for any fixed $y \in V$ by the C. Neumann series

$$u = \sum_{n=0}^{\infty}(-1)^n T^n g^V(\cdot,y),$$

where I is the identity operator. In fact, by using the 3G inequality (4.4) and (4.6) we infer that

$$|(Tg^V(\cdot,y))(x)| \leqq (|T|g^V(\cdot,y))(x) = \int_V g^V(x,z)g^V(z,y)d|\mu|(z)$$

$$\leqq \gamma(c)g^V(x,y)\int_V (N(x-z)+N(z-y))d|\mu(z)| \leqq qg^V(x,y)$$

on V, i.e. $|Tg^V(\cdot,y)| \leqq |T|g^V(\cdot,y) \leqq qg^V(\cdot,y)$ on V. Assuming the existence of $T^jg^V(\cdot,y)$ and $|T^jg^V(\cdot,y)| \leqq q^jg^V(\cdot,y)$ on V as the induction assumption, we deduce the existence of $T^{j+1}g^V(\cdot,y) = T(T^jg^V(\cdot,y))$ and

$$\begin{aligned}|T^{j+1}g^V(\cdot,y)| &= |T(T^jg^V(\cdot,y))| \leqq |T||T^jg^V(\cdot,y)| \leqq |T|(q^jg^V(\cdot,y)) \\ &= q^j|T|g^V(\cdot,y) \leqq q^j(qg^V(\cdot,y)) = q^{j+1}g^V(\cdot,y).\end{aligned}$$

By (4.3) we see that $T^jg^V(\cdot,y)$ is continuous on $\overline{V}\setminus\{y\}$ and vanishing on ∂V. Since the series converges uniformly on each compact subset of $\overline{V}\setminus\{y\}$ by the above observation, u is continuous on $\overline{V}\setminus\{y\}$ and

$$(1+q)^{-1}g^V(\cdot,y) \leqq u \leqq (1-q)^{-1}g^V(\cdot,y)$$

on V. Since $-ATu = u\mu$ and $-Ag^V(\cdot,y) = \delta_y$, δ_y being the Dirac delta at y, we derive from $u + Tu = g^V(\cdot,y)$ that $(-A+\mu)u = \delta_y$. Hence if we denote u by $g^V_\mu(\cdot,y) = g^V_{-A+\mu}(\cdot,y)$, we see that $g^V_\mu(\cdot,y)$ is the minimal positive continuous solution of

$$(-A+\mu)g^V_\mu(\cdot,y) = \delta_y \tag{4.7}$$

on V so that $g^V_\mu(\cdot,y)$ is, by definition, the μ *Green function* on V with its pole at y. By the above existence proof of $g^V_\mu(\cdot,y)$, we see that

$$(1+q)^{-1}g^V(\cdot,y) \leqq g^V_\mu(\cdot,y) \leqq (1-q)^{-1}g^V(\cdot,y) \tag{4.8}$$

on V and the validity of the so called *resolvent equation*

$$g^V_\mu(\cdot,y) = g^V(\cdot,y) - \int_V g^V(\cdot,z)g^V_\mu(z,y)d\mu(z). \tag{4.9}$$

We are ready to proceed to the proof of Reduction 3.2.

PROOF OF REDUCTION 3.2: Choose arbitrarily a member u in $S_\mu(M) = S_{-A+\mu}(M)$ and a point c in M. We have to show that u is mean continuous at c with respect to the metric $ds = a_{ij}(x)dx^idx^j$. Choose a small geodesic ball $V := B(c,r)$ satisfying (4.6). Then the lower semicontinuity of u implies that u is bounded from below on $\overline{V}$ so that there exist an $h \in H_\mu(V)$ and a Borel measure ν on $\overline{V}$ such that

$$u = h + g^V_\mu(\cdot,\nu), \tag{4.10}$$

where the μ Green potential on V with the measure ν is denoted by

$$g_\mu^V(\cdot,\nu) := \int_V g_\mu^V(\cdot,z)d\nu(z).$$

The relation (4.10) is a local form of the Riesz decomposition theorem and it is not difficult to give a direct analytic proof to it. But here we only quote the well known work of R.-M. Hervé [4] (see also [2], [8]). By integrating both sides of (4.9) over V by the measure ν and then by using the Fubini theorem we obtain

$$g_\mu^V(\cdot,\nu) = g^V(\cdot,\nu) - \int_V g^V(\cdot,z)g_\mu^V(z,\nu)d\mu(z). \tag{4.11}$$

Consider the measure ω given by $d\omega(z) = g_\mu^V(z,\nu)d\mu(z)$ and let $\omega = \omega^+ - \omega^-$ and $\mu = \mu^+ - \mu^-$ be Jordan decompositions of ω and μ, respectively. Then $d\omega^\pm(z) = g_\mu^V(z,\nu)d\mu^\pm(z)$ and, ω^+ and ω^- are Borel measures on V. From (4.11) it follows that

$$g_\mu^V(\cdot,\nu) = g^V(\cdot,\nu) - g^V(\cdot,\omega^+) + g^V(\cdot,\omega^-)$$

and therefore by (4.10) we have

$$u = h + g^V(\cdot,\nu) - g^V(\cdot,\omega^+) + g^V(\cdot,\omega^-). \tag{4.12}$$

Since $h \in C(V) \subset \mathrm{mc}(V, ds)$, we see that $h \in \mathrm{mc}(V, ds)$. As $-A$ Green potentials, each of $g^V(\cdot,\nu)$, $g^V(\cdot,\omega^+)$ and $g^V(\cdot,\omega^-)$ belongs to $S_{-A}(V, ds)$. By the assumption of our reduction, $S_{-A}(M) \subset \mathrm{mc}(M, ds)$ and finally we conclude that $u \in \mathrm{mc}(M)$ so that we can conclude by (4.12) that $S_{-A+\mu}(M) \subset \mathrm{mc}(M, ds)$. □

5. Proof of Theorem 3.1. Before proceeding to the proof of Theorem 3.1 we recall certain estimates of ratios of two different Green functions for different differential operators. We keep on denoting the coordinate of $x \in \mathbb{R}^d$ by $x = (x^1, \cdots, x^d)$. Let $\mathbb{D}$ be the open unit ball $\{x \in \mathbb{R}^d : |x| < 1\}$ so that $t\mathbb{D}$ $(t > 0)$ is the open ball with radius t centered at the origin 0. We denote by $\mathcal{L}(\kappa, \alpha)$ with constants $\kappa \in [1, \infty)$ and $\alpha \in (0, 1)$ the totality of differential operators L on $\mathbb{R}^d$ given by

$$Lu(x) := a^{ij}(x)\frac{\partial^2 u(x)}{\partial x^i \partial x^j} + b^i(x)\frac{\partial u(x)}{\partial x^i}, \tag{5.1}$$

where the coefficients a^{ij} and b^i of class C^∞ satisfy the following three conditions:
(i) For all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ we have

$$\sum_{i,j}^{1,\cdots,d} a^{ij}(x)\xi^i\xi^j \geqq (1/\kappa)|\xi|^2; \tag{5.2}$$

(ii) For all $x, y \in \mathbb{R}^d$ we have

$$\sum_{i,j}^{1,\cdots,d} |a^{ij}(x) - a^{ij}(y)| + \sum_{i}^{1,\cdots,d} |b^i(x) - b^i(y)| \leqq \kappa |x-y|^\alpha; \tag{5.3}$$

(iii) For all $x \in \mathbb{R}^d$ we have

$$\sum_{i,j}^{1,\cdots,d} |a^{ij}(x)| + \sum_{i}^{1,\cdots,d} |b^i(x)| \leqq \kappa. \tag{5.4}$$

The usual Laplace operator Δ (i.e. L with $a^{ij}(x) = \delta^{ij}$ (the Kronecker delta) and $b^i(x) \equiv 0$) is a typical member belonging to $\mathcal{L}(\kappa, \alpha)$ for $\kappa = d$ and any admissible α. We define the distance $\|L - \Delta\|$ between the operator $L \in \mathcal{L}(\kappa, \alpha)$ in (5.1) and $\Delta \in \mathcal{L}(\kappa, \alpha)$ for some fixed κ and α by

$$\|L - \Delta\| := \sup_{x \in \mathbb{R}^d} \left(\sum_{i,j}^{1,\cdots,d} |a^{ij}(x) - \delta^{ij}| + \sum_{i}^{1,\cdots,d} |b^i(x)| \right). \tag{5.5}$$

We denote by $g_L^{\mathbb{D}}(x,y)$ the Green function on $\mathbb{D}$ for the operator L in (5.1) and consider the quantity $C(L, \Delta) \in [1, +\infty]$ given by

$$C(L,\Delta) := \inf \left\{ C \in [1, +\infty] : C^{-1} \leqq \frac{g_L^{\mathbb{D}}(x,y)}{g_\Delta^{\mathbb{D}}(x,y)} \leqq C \quad \text{for all } x, y \in \mathbb{D} \right\}. \tag{5.6}$$

Then we have the following result due to H. Hueber and Sieveking (cf. [6] and [7]): $C(L, \Delta) < +\infty$ for every $L \in \mathcal{L}(\kappa, \alpha)$ with respect to any admissible κ and α; if $\Delta \in \mathcal{L}(\kappa, \alpha)$, then

$$\lim_{L \in \mathcal{L}(\kappa,\alpha), \|L - \Delta\| \to 0} C(L, \Delta) = 1. \tag{5.7}$$

Consider a local parameter $z = (z^1, \cdots, z^d)$ and a point $c \in M$, $z(c) = 0$. We say that z is a *geodesic coordinate* at c for (M, ds) if the metric tensor $(a_{ij}(z))$ corresponding to z satisfies the following

$$a_{ij}(z) - \delta_{ij} \in (z^1, \cdots, z^d)^2 \quad (i, j = 1, \cdots, d), \tag{5.8}$$

where $(z^1, \cdots, z^d)^2$ is the ideal generated by local functions $(z^i)^2$ $(i = 1, \cdots, d)$. Since $(a^{ij}(z)) = (a_{ij}(z))^{-1}$, we can easily verify the same property as (5.8) for the matrix $(a^{ij}(z))$ corresponding to z:

$$a^{ij}(z) - \delta^{ij} \in (z^1, \cdots, z^d)^2 \quad (i, j = 1, \cdots, d). \tag{5.9}$$

In the sequel we consider z mainly in small vicinities of c and therefore we may assume that z is valid in the ball $4\mathbb{D}$. We rewrite the operator A, for which the relation $S_{-A}(M) \subset \mathrm{mc}(M, ds)$ will be shown, as follows:

$$Au(z) = a^{ij}(z)\frac{\partial^2 u(z)}{\partial z^i \partial z^j} + B^i(z)\frac{\partial u(z)}{\partial z^i}, \tag{5.10}$$

where

$$B^i(z) = \frac{1}{\sqrt{a(z)}}\frac{\partial}{\partial z^j}\left(\sqrt{a(z)}a^{ij}(z)\right) + b^i(z) \quad (i = 1, \cdots, d) \tag{5.11}$$

are also of class C^∞ like $a^{ij}(z)$. Suppose u is a solution of $Au = 0$ on $r\overline{\mathbb{D}} := \overline{r\mathbb{D}}$. By the change of variables $z = rZ$ in (5.10) we see that

$$a^{ij}(rZ)\frac{\partial^2 u(rZ)}{\partial(rZ^i)\partial(rZ^j)} + B^i(rZ)\frac{\partial u(rZ)}{\partial(rZ^i)} = 0$$

on $\mathbb{D}$ and thus, by putting $U_r(Z) := u(rZ)$, we have

$$A_r U_r(Z) := a^{ij}(rZ)\frac{\partial^2 U_r(Z)}{\partial Z^i \partial Z^j} + rB^i(rZ)\frac{\partial U_r(Z)}{\partial Z^i} = 0,$$

i.e. $U_r(Z)$ is a solution of $A_r U_r = 0$ on $\overline{\mathbb{D}}$. Similarly, if u is a solution of $\Delta u = 0$ on $r\overline{\mathbb{D}}$, then $U_r(Z) := u(rZ)$ is a solution of $\Delta_r U_r = 0$ on $\overline{\mathbb{D}}$, where $\Delta_r = \Delta$. Even if we are considering that $\Delta_r = \Delta$ is defined only on $4\mathbb{D}$, it is naturally extended to the operator $\widehat{\Delta}_r = \Delta$ on $\mathbb{R}^d$ and $\widehat{\Delta}_r \in \mathcal{L}(d, \alpha)$ for any $\alpha \in (0,1)$. The operator A_r is also valid on $4\mathbb{D}$. Take a function $\varphi_r \in C^\infty([0, +\infty))$ such that $0 \leqq \varphi_r(t) \leqq 1$ on $[0, +\infty)$, $\varphi_r(t) = 1$ $(0 \leqq t \leqq 2)$, and $\varphi_r(t) = 0$ $(3 \leqq t < +\infty)$. Then consider the operator $\widehat{A}_r$ given by

$$\widehat{A}_r := \varphi_r(|Z|)A_r + (1 - \varphi(|Z|))\widehat{\Delta}_r.$$

By the construction we see that there are $\overline{\kappa} \geqq 1$ and $\overline{\alpha} \in (0,1)$ such that $\widehat{A}_r \in \mathcal{L}(\overline{\kappa}, \overline{\alpha})$ for every $0 < r < 1$. Let us stress once more that the most important point here is that $\widehat{A}_r \in \mathcal{L}(\overline{\kappa}, \overline{\alpha})$ holds for every $0 < r < 1$ so that $\overline{\kappa}$ and $\overline{\alpha}$ does not depend on $0 < r < 1$. Thus by setting $\kappa = \max(d, \overline{\kappa})$ and taking any $\alpha \in (0,1)$ we see that

$$\widehat{\Delta}_r,\ \widehat{A}_r \in \mathcal{L}(\kappa, \alpha) \quad (0 < r < 1). \tag{5.12}$$

We denote by $g^{\mathbb{D}}_{\widehat{\Delta}_r}(Z, Y)$ ($g^{\mathbb{D}}_{\widehat{A}_r}(Z, Y)$, resp.) the Green function on $\mathbb{D}$ for the operator $\widehat{\Delta}_r$ ($\widehat{A}_r$, resp.). Observe that, in view of (5.8), we have

$$\|\widehat{A}_r - \widehat{\Delta}_r\| = \sup_{z \in r\mathbb{D}}\left(\sum_{i,j}^{1,\cdots,d} |a^{ij}(z) - \delta^{ij}| + \sum_{i}^{1,\cdots,d} r|B^i(z)|\right) \to 0 \ (r \downarrow 0)$$

and then we see that, by (5.7), $\|\widehat{A}_r - \widehat{\Delta}_r\| \to 0$ $(r \downarrow 0)$ implies that

$$\lim_{r\downarrow 0} C(\widehat{A}_r, \widehat{\Delta}_r) = 1. \tag{5.13}$$

Let $g^{r\mathbb{D}}_{\Delta}(z,w)$ ($g^{r\mathbb{D}}_{A}(z,w)$, resp.) be the Green function on $r\mathbb{D}$ for the operator Δ (A, resp.). Then $g^{r\mathbb{D}}_{\Delta}(rZ,rW) = \alpha_r g^{\mathbb{D}}_{\widehat{\Delta}_r}(Z,W)$ and $g^{r\mathbb{D}}_{A}(rZ,rW) = \beta_r g^{\mathbb{D}}_{\widehat{A}_r}(Z,W)$ for some constants α_r and β_r. However, by (4.2), we must conclude $\alpha_r = \beta_r$ so that

$$g^{r\mathbb{D}}_{A}(rZ,rW)/g^{r\mathbb{D}}_{\Delta}(rZ,rW) = g^{\mathbb{D}}_{\widehat{A}_r}(Z,W)/g^{\mathbb{D}}_{\widehat{\Delta}_r}(Z,W) \quad (Z,W \in \mathbb{D}).$$

The right hand side of the above formula belongs to the interval

$$(C(\widehat{A}_r, \widehat{\Delta}_r)^{-1}, C(\widehat{A}_r, \widehat{\Delta}_r)).$$

Hence, by (5.13), we have the following: for any $C \in (1,+\infty)$ there is an $R = R_C \in (0,1)$ such that

$$C^{-1} g^{r\mathbb{D}}_{\Delta}(z,y) \leqq g^{r\mathbb{D}}_{A}(z,y) \leqq C g^{r\mathbb{D}}_{\Delta}(z,y) \quad (x,y \in r\overline{\mathbb{D}},\ r \in (0,R)). \tag{5.14}$$

By $a(0) = \det(a_{ij}(0)) = \det(\delta_{ij}) = 1$, we can also assume that

$$C^{-1} \leqq \sqrt{a(z)} \leqq C \quad (z \in r\overline{\mathbb{D}},\ r \in (0,R)). \tag{5.15}$$

PROOF OF THEOREM 3.1: Take an arbitrarily chosen $u \in S_{-A}(M)$ and pick any point $c \in M$. We fix a geodesic coordinate $z = (z^1, \cdots, z^d)$, $z(c) = 0$, valid in $4\mathbb{D}$. We first prove

$$\lim_{r\downarrow 0} \frac{1}{\lambda(r\mathbb{D})} \int_{r\mathbb{D}} u(z)d\lambda(z) = u(c). \tag{5.16}$$

Since u is lower semicontinuous on $\overline{\mathbb{D}}$, we can assume that $u > 0$ on $\mathbb{D}$. If $u(c) = +\infty$, again by the lower semicontinuity of u, the validity of (5.16) can be easily deduced. Hence we only have to prove (5.16) under the assumption that $u > 0$ on $\mathbb{D}$ and $u(c) < +\infty$. Once again by the lower semicontinuity of u there is a $\delta \in (0,1)$ for any $\varepsilon > 0$ given in advance such that $u(z) \geqq u(c) - \varepsilon$ for every $z \in \delta\mathbb{D}$. Thus

$$\int_{r\mathbb{D}} u(z)d\lambda(z) \geqq (u(c) - \varepsilon)\lambda(r\mathbb{D})$$

for every $r \in (0,\delta)$ and therefore we infer that

$$\liminf_{r\downarrow 0} \frac{1}{\lambda(r\mathbb{D})} \int_{r\mathbb{D}} u(z)d\lambda(z) \geqq u(c). \tag{5.17}$$

Choose any $C \in (1, +\infty)$ and take the corresponding $R = R_C \in (0,1)$ in (5.14). For each $r \in (0, R)$, let $(H_{-A})_u^{r\mathbb{D}}$ be the PWB (i.e. Perron-Wiener-Brelot) solution to the Dirichlet problem on $r\mathbb{D}$ with boundary data u on $\partial(r\mathbb{D})$ corresponding to the operator A and also $(H_{-\Delta})_u^{r\mathbb{D}}$ be the similar one as $(H_{-A})_u^{r\mathbb{D}}$ corresponding to the operator Δ. We are to compare $v_1 := (H_{-\Delta})_u^{r\mathbb{D}}$ with $v_2 := (H_{-A})_u^{r\mathbb{D}}$. Choose an increasing sequence $(r_n)_{n\in\mathbb{N}}$ in $(0, r)$ with $r_n \uparrow r$ $(n \uparrow +\infty)$. Here $\mathbb{N}$ is the totality of positive integers. Construct a function p_n on $r\mathbb{D}$ for each $n \in \mathbb{N}$ such that $p_n = v_1$ on $r_n\mathbb{D}$ and $p_n \in C(r\overline{\mathbb{D}} \setminus r_n\mathbb{D}) \cap H_{-\Delta}(r\mathbb{D} \setminus r_n\overline{\mathbb{D}})$ with boundary values $p_n|\partial(r\mathbb{D}) = 0$ and $p_n|\partial(r_n\mathbb{D}) = v_1$. Clearly p_n is a $-\Delta$ potential on $r\mathbb{D}$ (i.e. $p_n \in S_{-\Delta}(r\mathbb{D})$ whose greatest $-\Delta$ harmonic minorant on $r\mathbb{D}$ is zero). Therefore, by the Riesz decomposition theorem, there exists a Borel measure ν_n supported by $\partial(r_n\mathbb{D})$ such that

$$p_n(z) = \int g_\Delta^{r\mathbb{D}}(z, y)d\nu_n(y) \quad (z \in r\mathbb{D})$$

for each $n \in \mathbb{N}$. By the construction, $p_n(z) \uparrow v_1(z)$ $(n \uparrow +\infty)$. Corresponding to $p_n(z)$ we define

$$q_n(z) = \int g_A^{r\mathbb{D}}(z, y)d\nu_n(y) \quad (z \in r\mathbb{D}),$$

which belongs to $H_{-A}(r\mathbb{D} \setminus \partial(r_n\mathbb{D})) \subset H_{-A}(r_n\mathbb{D})$. By (5.14) we have

$$C^{-1}p_n(z) \leqq q_n(z) \leqq Cp_n(z) \quad (z \in r\mathbb{D},\ n \in \mathbb{N})$$

and in particular $q_n(z) \leqq Cv_1(z)$ $(z \in r\mathbb{D})$ for every $n \in \mathbb{N}$. Hence the family $\{q_n : n \in \mathbb{N}\}$ of functions q_n on $r\mathbb{D}$ is uniformly bounded and equicontinuous as a result of the Harnack inequality so that the family $\{q_n : n \in \mathbb{N}\}$ is normal. Hence by replacing $(r_n)_{n\in\mathbb{N}}$ by its suitable subsequence if necessary we can assume that the sequence $(q_n)_{n\in\mathbb{N}}$ is locally uniformly convergent on $r\mathbb{D}$ and therefore

$$q(z) := \lim_{n\to\infty} q_n(z) \quad (z \in r\mathbb{D})$$

is $-A$ harmonic on $r\mathbb{D}$. Hence

$$C^{-1}v_1(z) \leqq q(z) \leqq Cv_1(z) \quad (z \in r\mathbb{D}).$$

From this it follows that

$$C^{-1}v_2(z) \leqq q(z) \leqq Cv_2(z) \quad (z \in r\mathbb{D})$$

and finally we can conclude that

$$C^{-2}(H_{-\Delta})_u^{r\mathbb{D}} \leqq (H_{-A})_u^{r\mathbb{D}} \leqq C^2(H_{-\Delta})_u^{r\mathbb{D}} \quad (z \in r\mathbb{D}). \tag{5.18}$$

In order to complete the proof of (5.16) we need to establish the reversed assertion to (5.17) so that we now prove

$$\limsup_{r\downarrow 0}\frac{1}{\lambda(r\mathbb{D})}\int_{r\mathbb{D}}u(z)d\lambda(z)\leqq u(c). \tag{5.19}$$

We fix an arbitrary $r\in(0,R_C)$. Based upon the $-A$ harmonical concaveness of u, the relation (5.18), and the Gauss mean value theorem, we infer that

$$u(c)\geqq (H_{-A})_u^{t\mathbb{D}}(c)\geqq C^{-2}\cdot(H_{-\Delta})_u^{t\mathbb{D}}(c)=C^{-2}\cdot\frac{1}{\sigma}\int_{\partial(t\mathbb{D})}u(z)d\sigma(z)$$

for every $0<t\leqq r$, where $d\sigma$ and σ are Euclidean area element on and Euclidean area of the unit sphere $\partial\mathbb{D}$ in $\mathbb{R}^d$. Hence we have, by Fubini theorem,

$$\begin{aligned}\frac{r^d}{d}u(c)&=\int_0^r u(c)t^{d-1}dt\geqq\int_0^r\left(\frac{C^{-2}}{\sigma}\int_{\partial(t\mathbb{D})}u(z)d\sigma(z)\right)t^{d-1}dt\\&=\frac{C^{-2}}{\sigma}\int_{[0,r]\times\partial(t\mathbb{D})}u(z)t^{d-1}dtd\sigma(z)=\frac{C^{-2}}{\sigma}\int_{r\mathbb{D}}u(z)d\lambda_e(z),\end{aligned}$$

where λ_e is the Euclidean volume measure on $\mathbb{R}^d$, i.e. $d\lambda_e(z)=dz^1\cdots dz^d$. Since $r^d(\sigma/d)=\lambda_e(r\mathbb{D})$, we see that

$$\frac{1}{\lambda_e(r\mathbb{D})}\int_{r\mathbb{D}}u(z)d\lambda_e(z)\leqq C^2u(c)\quad(r\in(0,R)).$$

By applying (5.15) to $d\lambda(z)=\sqrt{a(z)}d\lambda_e(z)$, the above inequality implies that

$$\frac{1}{\lambda(r\mathbb{D})}\int_{r\mathbb{D}}u(z)d\lambda(z)\leqq C^4u(c)\quad(r\in(0,R)).$$

Therefore we conclude that

$$\limsup_{r\downarrow 0}\frac{1}{\lambda(r\mathbb{D})}\int_{r\mathbb{D}}u(z)d\lambda(z)\leqq C^4u(c).$$

Since $C\in(1,+\infty)$ is arbitrary, on letting $C\downarrow 1$ in the above inequality we deduce (5.19). We have thus established (5.16).

To finish the proof of Theorem 3.1, we have to replace $r\mathbb{D}$ by the geodesic ball $B(c,r)$ in (5.16). Thus our final task is to compare the size of the geodesic ball $B(c,r)$ of radius r centered at c with that of the Euclidean ball $r\mathbb{D}$ of radius r centered at c, where $(\mathbb{D},z)$ is the geodesic coordinate with $z(c)=0$, for sufficiently small $r>0$. Thus we compare the geodesic distance $s=s(z)$ between the center c of $\mathbb{D}$, i.e. $z(c)=0$, and the point $z\in\mathbb{D}$ enough close to 0 with the Euclidean distance $|z|=\sqrt{(z^1)^2+\cdots+(z^d)^2}$ between 0 and $z\in\mathbb{D}$, where $(\mathbb{D},z)$, $z=$

$(z^1, \cdots, z^d)$, is the geodesic coordinate in (M, ds). Recall that geodesic lines are given in terms of geodesic coordinates z^i $(i = 1, \cdots, d)$ as follows:

$$\frac{d^2 z^i}{ds^2} + \sum_{i,j} \Gamma_j{}^i{}_k \frac{dz^j}{ds}\frac{dz^k}{ds} = 0 \tag{5.20}$$

with conditions $(z^i)_0 = 0$ and $dz^i/ds = \xi^i$; here $(\Gamma_j{}^i{}_k)_0 = 0$ and $\sum_i (\xi^i)^2 = 1$, where $\Gamma_j{}^i{}_k$ is the Christoffel symbol of the second kind. From (5.20) we can derive the relation

$$|z| - s(z) = O(s(z)^3), \tag{5.21}$$

where $O(t)$ is the Landau oh so that $O(t)/t$ is bounded for small t. Observe that $|z| \to 0$ when and only when $s(z) \to 0$. Hence, in particular, we see that

$$s(z) - s(z)^2 \leqq |z| \leqq s(z) + s(z)^2 \tag{5.22}$$

for every $z \in \mathbb{D}$ sufficiently close to the origin 0, i.e. c. Hence for all small $r > 0$ we have the inclusion relations

$$(r - r^2)\mathbb{D} \subset B(c, r) \subset (r + r^2)\mathbb{D}. \tag{5.23}$$

We are ready to proceed the final job: evaluating the mean

$$I_r := \frac{1}{\lambda(B(c,r))} \int_{B(c,r)} u(z) d\lambda(z).$$

By using (5.23) we first evaluate I_r from above as

$$I_r \leqq \frac{\lambda((r+r^2)\mathbb{D})}{\lambda((r-r^2)\mathbb{D})} \left(\frac{1}{\lambda((r+r^2)\mathbb{D})} \int_{(r+r^2)\mathbb{D}} u(z) d\lambda(z) \right) \tag{5.24-1}$$

and then similarly from below as

$$I_r \geqq \frac{\lambda((r-r^2)\mathbb{D})}{\lambda((r+r^2)\mathbb{D})} \left(\frac{1}{\lambda((r-r^2)\mathbb{D})} \int_{(r-r^2)\mathbb{D}} u(z) d\lambda(z) \right). \tag{5.24-2}$$

Observe that for $t > 0$

$$\lambda(t\mathbb{D}) = \int_{t\mathbb{D}} \sqrt{a(z)} dz^1 \cdots dz^d, \quad \lambda_e(t\mathbb{D}) = \int_{t\mathbb{D}} dz^1 \cdots dz^d = t^d(\sigma/d).$$

By virtue of that $\sqrt{a(z)} \to \sqrt{a(0)} = 1$ $(z \to 0)$, we infer that $\lambda(t\mathbb{D})/\lambda_e(t\mathbb{D}) \to \sqrt{a(0)} = 1$ $(t \downarrow 0)$ and therefore

$$\frac{\lambda((r+r^2)\mathbb{D})}{\lambda((r-r^2)\mathbb{D})} = \frac{\lambda((r+r^2)\mathbb{D})}{\lambda_e((r+r^2)\mathbb{D})} \cdot \frac{\lambda_e((r-r^2)\mathbb{D})}{\lambda((r-r^2)\mathbb{D})} \cdot \frac{(r+r^2)^d(\sigma/d)}{(r-r^2)^d(\sigma/d)} \to 1 \quad (r \downarrow 0).$$

Thus on letting $r \downarrow 0$ in (5.24-1) and (5.24-2) we deduce by (5.16) that

$$u(c) \leqq \liminf_{r\downarrow 0} I_r \leqq \limsup_{r\downarrow 0} I_r \leqq u(c),$$

which amounts to the same as

$$\lim_{r\downarrow 0} \frac{1}{\lambda(B(c,r))} \int_{B(c,r)} u(z)d\lambda(z) = u(c).$$

This means that $S_{-A}(M) \subset \mathrm{mc}(M, ds)$. This with Reduction 3.2 implies that $S_{-A+\mu}(M) \subset \mathrm{mc}(M, ds)$. The proof of Theorem 3.1 is herewith complete. □

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